Quantum Hairs and Isolated Horizon Entropy from Chern-Simons Theory

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We articulate the fact that the loop quantum gravity description of the quantum states of black hole horizons, modeled as Quantum Isolated Horizons (QIHs), is completely characterized in terms of two independent integer-valued quantum 'hairs', viz,. the coupling constant of the quantum $SU(2)$ Chern Simons theory describing QIH dynamics, and the number of punctures produced by the bulk spin network edges piercing the isolated horizon (which act as pointlike sources for the Chern Simons fields). We demonstrate that the microcanonical entropy of macroscopic (both parameters assuming very large values) QIHs can be obtained directly from the microstates of this Chern-Simons theory, using standard statistical mechanical methods, without having to additionally postulate the horizon as an ideal gas of punctures, or incorporate any additional classical or semi-classical input from general relativity vis-a-vis the functional dependence of the IH mass on its area, or indeed, without having to restrict to any special class of spins. Requiring the validity of the Bekenstein-Hawking area law relates these two parameters (as an equilibrium 'equation of state') and consequently imposes restrictions on the allowed values of the Barbero-Immirzi parameter. The logarithmic correction to the area law obtained a decade ago by R. Kaul and one of us (P.M.), ensues straightforwardly, with precisely the coefficient $-3/2$, making it a signature of the loop quantum gravity approach to black hole entropy.

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I. INTRODUCTION

Perhaps the clearest description of the quantum states of generic (extremal or non-extremal) four dimensional black hole horizons in the presence of matter and/or radiation (which do not cross the horizons) is in terms of the quantization of the classical Isolated Horizon (IH) phase space which yields the kinematical Hilbert space for the Quantum Isolated Horizon (QIH) [1, 2] derived from loop quantum gravity (LQG). In this description, the physical states of a QIH are the gauge singlet states of an $SU(2)$ Chern Simons (CS) theory coupled to the punctures endowed with SU(2) spins deposited on the QIH by the intersecting edges of the bulk spin network describing the bulk quantum geometry. In other words, the QIH is taken to have the topology of a two-sphere with punctures carrying spins induced by the floating lattice known as the spin network whose edges carry spins. Such configurations are expected to arise as solutions of the full quantum dynamical equations describing bulk quantum geometry, in particular the quantum Hamiltonian constraint. Whether or not this actually happens remains a question for the future. In this situation, it is worthwhile to investigate properties of the kinematical Hilbert space to see how the issue of black hole entropy can be addressed.

There are a plethora of approaches to black hole entropy, many semiclassical ones with diverse claims for the black hole entropy having logarithmic corrections to the area law [3, 6–10], with various coefficients – some postive and some negative. In all these computations, one usually computes the entropy of quantum 'matter' fields (including gravitons) coupled to the classical background spacetime metric of a black hole. Even in approaches where the classical metric is 'integrated over', the functional integral over metrics is invariably saturated by the classical black hole metric. In other words, non-perturbatively large quantum fluctuations of the spacetime - which cannot meaningfully be separated into a classical background metric and its fluctuations - are usually ignored in these computations. Thus, these computations do *not* take

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into account the entirety of quantum states corresponding to quantum spacetime fluctuations of a black hole spacetime. Rather, they account for non-gravitational states *entangled* with the horizon, as discussed in ref. [11, 12]. In contrast, the LQG approach focuses on the quantum states describing the *quantum* geometry of the horizon, without assuming any entanglement with quantum matter states in its vicinity. The classical metric of the black hole plays no role in this approach. The LQG computation thus yields what one may call the *gravitational* (or 'spacetime') entropy of a black hole, as opposed to the non-gravitational entanglement entropy computed in other approaches, which depends rather directly on the classical black hole metric in all cases. Indeed, the two together would most likely constitute the total entropy of a quantal black hole spacetime, so that the results of both directions of computation are complimentary rather than competitively comparable. In other words, the entanglement entropy considered in the other approaches must be an additional contribution to the horizon entropy, over and above the spacetime entropy computed within the LQG approach.

The description of a QIH in terms of CS states coupled to bulk quantum geometry is itself a radical departure from a the general relativistic description of a classical event horizon. Of course, a QIH does not radiate or accrete matter/radiation, and can only be a part of a more complete description of a radiant black hole comprising of Trapping [18] or Dynamical [14] horizons. Recently, such horizons have been argued to emit Hawking radiation [15] with the standard black body spectrum, raising hopes that a deeper understanding of black holes within the LQG approach might be around the corner. However, there are semi-classical assumptions that appear to be necessary to supplement the premises of LQG, which manifest a lingering incompleteness in basic understanding of the quantum geometry of IHs.

The object of the present paper is to articulate a fact that has already been implicit in recent literature [19], however, *without* the extra semi-classically motivated assumptions used earlier : that a complete description of QIH is possible with two integer-valued quantum parameters ('hairs'), namely the CS coupling constant k related to the classical horizon area, and the number N of punctures to which the CS fields couple. Indeed, the microcanonical entropy can be obtained directly from the formula derived fourteen years ago by one of us (P. M.) together with R. K. Kaul [20] for the total number of $SU(2)$ singlet states coupled to a fixed set of spins $j_1, j_2, ..., j_N$, $j_i \in [1/2, k/2] \forall i = 1, ...N$. Summing this formula over all spins $\{j_i\}$ and using the Multinomial expansion of elementary algebra, this degeneracy can be expressed in terms of sums over spin *configurations*, i.e., number of punctures $n_j \in [0, N]$ for spins $j = 1/2, ..., k/2$. Using standard tenets of equilibrium statistical mechanics, one looks for the 'Most Probable Spin Configuration' by maximizing the microcanonical entropy subject to the constraints [21] of a fixed N and fixed large QIH area differing from the classical area only to $O(\ell_p^2)$, where ℓ_p is the Planck length. Requiring that the resulting formula for the microcanonical entropy of a macroscopic ($k \ggg 1,\ N \ggg 1$) QIH reproduce the Bekenstein-Hawking area law in terms of classical IH area A_{cl}, imposes a restriction on the Barbero-Immirzi parameter γ in the definition of $k \equiv A_{cl}/4\pi\gamma\ell_p^2$, that this must lie within a specific interval on the real line [22]. Subleading corrections to the area law ensue naturally from the derived formula, especially the leading logarithmic (in *classical* area) correction with coefficient $-3/2$ found longer than a decade ago by Kaul and Majumdar [23] for a dominant class of spins, as we shall see in the sequel. These results have also been rederived recently in ref. [24–27]

The important value-addition here is the direct link to the LQG formulation of QIHs in terms of $SU(2)$ CS states coupled to punctures carrying spin, without any trappings of a semi-classical nature depending on classical metrical properties, nor any restriction to a specific class of spin values at the punctures. While many of the extant papers in the recent literature on QIH entropy within the LQG approach simply count states of an ideal gas of spins, without much allusion to the underlying theory of QIHs in terms of quantum CS states, our approach here has been to underline the CS theory underpinning, and to make the argument as self-contained as possible within the original LQG approach pioneered by Ashtekar and coworkers. This paper has some overlap with the recent review of Kaul [28] as far as some parts of the calculations are concerned, but the emphasis on direct link-up with the CS theory perspective and some of the details are different and are, hopefully, of inherent merit.

II. PHYSICAL CHERN-SIMONS STATES OF THE QIH

The computation of the microcanonical entropy formula of Kaul and Majumdar [20] proceeds from the expression for the number of conformal blocks of $SU(2)_k$ WZW model on a 2-sphere with marked points

(punctures) carrying spin which, in the seminal work of Witten [29] has been shown to give the dimensionality of the singlet part of the Hilbert space of $SU(2)$ CS states on $\mathbf{R}\otimes S^2$ coupled to punctures on the S^2. Using this remarkable connection and the fusion algebra and the Verlinde formula, the degeneracy of the microstates is expressed as [20]

$$\Omega(j_1,\cdots,j_N) = \frac{2}{k+2}\sum_{a=1}^{k+1}\frac{\sin\frac{a\pi(2j_1+1)}{k+2}\cdots\sin\frac{a\pi(2j_N+1)}{k+2}}{\left(\sin\frac{a\pi}{k+2}\right)^{N-2}} \tag{1}$$

To obtain the total number of conformal blocks, this expression must be summed over all possible spin values at each puncture. Before doing the sum over each spin, let us write down the quantum mechanical wave function for a QIH as a linear combination of the basis states represented by the spin network basis. Thus, a generic normalized wave function for the QIH, in this spin network basis, can be written as

$$|\Psi\rangle = \sum_{j_1,\cdots,j_N}\tilde{c}(j_1,\cdots,j_N)\;|j_1,\cdots,j_N\rangle \tag{2}$$

where $\langle\Psi|\Psi\rangle = 1$, $j_i \in [1/2,k/2]\forall i = 1,...,N$. The latter requirement merely reiterates the fact that one must restrict to *unitary* representations of the $SU(2)_k$ current algebra. The total number of punctures N and the Chern-Simons level k being the two independent parameters of the theory [22], the number of microstates of a QIH characterized by given values of N and k can be written as

$$\Omega(N,k) = \sum_{j_1,\cdots,j_N} p(j_1,\cdots,j_N)\;\Omega(j_1,\cdots,j_N) \tag{3}$$

where $p(j_1,\cdots,j_N) = |\tilde{c}(j_1,\cdots,j_N)|^2$ is the probability of finding the QIH in a particular eigenstate configured by the spin sequence $(j_1,\cdots,j_N)$. It follows from the normalization condition that $\sum_{j_1,\cdots,j_N} p(j_1,\cdots,j_N) = 1$.

Now, since we will apply the method of most probable distribution to find the microcanonical entropy of the QIH, it is convenient to switch over to the configuration basis. At first we write the wave function of the QIH in the configuration basis as $|\Psi\rangle = \sum_{\{s_j\}} c[\{s_j\}]|\{s_j\}\rangle$. Then the number of microstates for a QIH with a given N and k will be obtained by summing over all possible configurations(eigenstates) and is written as

$$\Omega(N,k) = \sum_{\{s_j\}}\omega[\{s_j\}]\;\Omega[\{s_j\}] \tag{4}$$

where $\sum_{\{s_j\}}\omega[\{s_j\}] = \sum_{\{s_j\}}|c[\{s_j\}]|^2 = \langle\Psi|\Psi\rangle = 1$, $\omega[\{s_j\}]$ being the probability to find the QIH in the particular eigenstate designated by the spin configuration $\{s_j\}$ and

$$\Omega[\{s_j\}] = \frac{N!}{\prod_j s_j!}g[\{s_j\}] \tag{5}$$

where j runs from $1/2$ to $k/2$ as usual and

$$g[\{s_j\}] = \frac{2}{k+2}\sum_{a=1}^{k+1}\sin^2\frac{a\pi}{k+2}\prod_j\left\{\frac{\sin\frac{a\pi(2j+1)}{k+2}}{\sin\frac{a\pi}{k+2}}\right\}^{s_j} \tag{6}$$

The combinatorial factor in eq.(5) reflects the *statistical* distinguishablity of the punctures, a property inherited by the punctures in the quantization procedure of the classical IH due to the nontrivial holonomies of the Chern-Simons connection on the IH along disjoint closed loops about the punctures[2].

A. Multinomial Expansion : The Link

It is worth mentioning that, even though eq.(5) resembles the formula for the microstates of an ideal gas of spins obeying Maxwell-Boltzmann statistics, it has been derived directly from the formula (1) within the

present scenario of QIH. One should keep in mind that eq.(5) is just another form of eq.(3) being written in a different basis for convenience of the statistical formulation. Hence, for the sake of clarity, let us derive eq.(5) directly from eq.(3).

First of all, since the punctures do not *physically* interact with each other, we can write $p(j_1, \cdots, j_N) = p(j_1)\cdots p(j_N)$. Before proceeding further, let us note this crucial point in slightly more detail. Although there is no physical interaction among the punctures (i.e. the spin at a puncture can take any value irrespective of spin values at other punctures) only those configurations giving rise to SU(2) singlet states are physical, the others being unphysical 'gauge' states eliminated by the quantum Gauss law constraint. The number of singlet states for a given ordered sequence of spins $(j_1, \cdots, j_N)$ can be written as a linear combination of delta functions [20]. Now, the formula for the microstates given by eq.(1) is directly derived from this delta function formula using Fourier transform. On the other hand, one can treat this delta function formula as a constraint which excludes the counting the extra 'gauge' degrees of freedom. Thus, in some sense, the spins at different punctures must be correlated to give rise to the physical states of the QIH. In fact this is the reason why the second order corrections to the entropy i.e. the logarithmic corrections [23] are not additive, which has been correctly pointed out in [30]. Hence, considering $p(j_1, \cdots, j_N) = p(j_1)\cdots p(j_N)$ is completely justified since the delta function formula being a constraint itself takes care of this virtual correlation among the punctures.

Hence, using eq.(1) one can write eq.(3) in the following explicit form

$$\begin{aligned}\Omega(N,k) &= \frac{2}{k+2}\sum_{a=1}^{k+1}\sin^2\frac{a\pi}{k+2}\sum_{j_1,\cdots,j_N}\prod_{r=1}^{N}\left\{p(j_r)\frac{\sin\frac{(2j_r+1)a\pi}{k+2}}{\sin\frac{a\pi}{k+2}}\right\}\\ &= \frac{2}{k+2}\sum_{a=1}^{k+1}\sin^2\frac{a\pi}{k+2}\prod_{r=1}^{N}\sum_{j_1,\cdots,j_N}\left\{p(j_r)\frac{\sin\frac{(2j_r+1)a\pi}{k+2}}{\sin\frac{a\pi}{k+2}}\right\}\\ &= \frac{2}{k+2}\sum_{a=1}^{k+1}\sin^2\frac{a\pi}{k+2}\left[\sum_{j}\left\{p(j)\frac{\sin\frac{(2j+1)a\pi}{k+2}}{\sin\frac{a\pi}{k+2}}\right\}\right]^N\end{aligned}$$

Using Multinomial expansion, the above expression can be recast into the following form

$$\begin{aligned}\Omega(N,k) &= \frac{2}{k+2}\sum_{a=1}^{k+1}\sin^2\frac{a\pi}{k+2}\sum_{\{s_j\}}\frac{N!}{\prod_j s_j!}\prod_j\left\{p(j)\frac{\sin\frac{(2j+1)a\pi}{k+2}}{\sin\frac{a\pi}{k+2}}\right\}^{s_j}\\ &= \sum_{\{s_j\}}\omega[\{s_j\}]\left[\frac{2}{k+2}\sum_{a=1}^{k+1}\sin^2\frac{a\pi}{k+2}\frac{N!}{\prod_j s_j!}\prod_j\left\{\frac{\sin\frac{(2j+1)a\pi}{k+2}}{\sin\frac{a\pi}{k+2}}\right\}^{s_j}\right]\end{aligned} \tag{7}$$

where $\omega[\{s_j\}] = \{p(j)\}^{s_j}$. This is exactly eq.(4), which is nothing but eq.(3) written in the configuration basis.

III. QUANTUM HAIRS AND MICROCANONICAL ENSEMBLE

The next task is to define the microcanonical ensemble of QIHs by identifying and fixing the parameters of the theory. Right at the beginning, it has already been mentioned that k and N are the two independent parameters, whose values, once specified, determine the number of microstates of a QIH given by the eq.(7). Thus k and N completely characterize a QIH. To obtain the microcanonical entropy, one must maximize its expression with respect to the spin configurations, to determine the most probable spin configuration, subject to the restriction that the QIH area must equal the classical area upto $O(l_P^2)$ and the number N of punctures is fixed.

The area eigenvalue equation for a particular eigenstate of the QIH in the configuration basis can be written as

$$\hat{A}|\{s_j\}\rangle = 8\pi\gamma\ell_p^2\sum_j s_j\sqrt{j(j+1)}\,|\{s_j\}\rangle \tag{8}$$

Hence, the expectation value of the area operator for the QIH is given by

$$\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle \ = \ 8\pi\gamma\ell_p^2 \sum_{\{s_j\}} \omega[\{s_j\}] \sum_j s_j \sqrt{j(j+1)} \ = \ A_{cl} \pm O(\ell_p^2) \tag{9}$$

where A_{cl} is the area of the classical IH closely represented by the QIH. Scaling the equation by $8\pi\gamma\ell_p^2$ and using $\langle \hat{A} \rangle / 8\pi\gamma\ell_p^2 \approx A_{cl}/8\pi\gamma\ell_p^2 = k/2$ [2, 16], we obtain

$$\sum_{\{s_j\}} \omega[\{s_j\}] \sum_j s_j \sqrt{j(j+1)} = \frac{k}{2} \tag{10}$$

In this process we can also avoid the involvement of the ambiguous parameter γ which will be ultimately fixed at the end by the usual argument of the validity of the BHAL.

Apart from this, the expectation value of the number of punctures for the QIH is given by

$$\langle \hat{N} \rangle = \langle \Psi | \hat{N} | \Psi \rangle = \sum_{\{s_j\}} \omega[\{s_j\}] \sum_j s_j \ = \ N \tag{11}$$

where $\hat{N}$ can be considered as the operator for the number of punctures for a QIH. One should note that unlike the case of area the expectation value of $\hat{N}$ is *exactly* equal to the total number of punctures N. Now, since we are doing equilibrium statistical mechanics (thus stability is assumed) of QIH with large area ($A_{cl} \gg \ell_p^2$) and large number of punctures ($N \gg 1$), for all practical purposes, we can neglect the fluctuations and also consider that the dominant contribution to the entropy comes from a most probable configuration, $\{s_j^\star\}$ i.e. $\omega[\{s_j^\star\}] \simeq 1$. Hence, the spin configuration $\{s_j^\star\}$ must obey the following constraints

$$\mathcal{C}_1 : \sum_j s_j^\star = N \tag{12a}$$

$$\mathcal{C}_2 : \sum_j s_j^\star \sqrt{j(j+1)} = \frac{k}{2} \tag{12b}$$

Hence, we define a microcanonical ensemble of QIHs by assigning fixed values of k and N respectively. The obvious following step is the computation of the microcanonical entropy of a QIH characterized by fixed values of k and N.

IV. MICROCANONICAL ENTROPY

Having defined the microcanonical ensemble appropriately, we shall now derive the microcanonical entropy of a QIH. The microcanonical entropy of a QIH for given values of k and N is written as

$$S_{MC} = \log \Omega(N, k) \simeq \log \Omega[\{s_j^\star\}] \tag{13}$$

where we have set the Maxwell-Boltzmann constant to unity. Variation of $\log \Omega[\{s_j\}]$ with respect to s_j, subject to the constraints $\mathcal{C}_1$ and $\mathcal{C}_2$, yields the distribution function for the most probable configuration $\{s_j^\star\}$ which maximizes the entropy of the QIH. In other words, $s_j^\star$ satisfies the variational equation written as

$$\delta \log \Omega[\{s_j\}] - \sigma \sum_j \delta s_j - \lambda \sum_j \delta s_j \sqrt{j(j+1)} = 0 \tag{14}$$

where δ represents variation with respect to s_j, σ and λ are the Lagrange multipliers for $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. This yields the most probable distribution given by

$$s_j^\star = N \exp\left[-\lambda \sqrt{j(j+1)} - \sigma + \frac{\delta}{\delta s_j} \log g[\{s_j\}] \right] \tag{15}$$

To proceed further we calculate $g[\{s_j\}]$ explicitly using saddle point approximation in the limit $k, N \to \infty$, which is appropriate for large black holes. First of all, we rewrite eq.(6) replacing the summation over a by integration as

$$\begin{aligned} g[\{s_j\}] &\simeq \frac{2}{k+2}\int_1^{k+1} \sin^2\frac{a\pi}{k+2}\prod_j\left\{\frac{\sin\frac{a\pi(2j+1)}{k+2}}{\sin\frac{a\pi}{k+2}}\right\}^{s_j} da \\ &= \frac{2}{\pi}\int_\epsilon^{\pi-\epsilon} \sin^2\theta\ \prod_j\left\{\frac{\sin(2j+1)\theta}{\sin\theta}\right\}^{s_j} d\theta \end{aligned} \tag{16}$$

where we have applied a change in the integration variable as $a\pi/(k+2) = \theta$ and for which the limits follow with $\epsilon = \pi/(k+2)$. Now, for $k \to \infty$, $\epsilon \to 0$. Hence, we can safely write

$$\begin{aligned} g[\{s_j\}] &\simeq \frac{2}{\pi}\int_0^{\pi} \sin^2\theta\ \prod_j\left\{\frac{\sin(2j+1)\theta}{\sin\theta}\right\}^{s_j} d\theta \\ &= \frac{1}{\pi}\int_0^\pi \exp\left[G(\theta,k)\right] d\theta - \frac{1}{\pi}\int_0^\pi \exp\left[\ln(\cos 2\theta) + G(\theta,k)\right] d\theta \end{aligned} \tag{17}$$

where $G(\theta,k) = \sum_j s_j \log\left\{\frac{\sin(2j+1)\theta}{\sin\theta}\right\}$. The above two integrations can be performed by the saddle point method. To begin with, it is straightforward to show that

$$\lim_{\theta_0\to 0} G'(\theta,k)|_{\theta_0} = 0 \tag{18a}$$

$$\lim_{\theta_0\to 0} \{-2\tan 2\theta + G'(\theta,k)\}|_{\theta_0} = 0 \tag{18b}$$

which implies that the saddle point $\theta_0 \simeq 0$ ($'$ denotes partial derivative with respect to θ). Now, Taylor expanding $G(\theta,k)$ about the saddle point θ_0 up to second order and applying the saddle point conditions from eqs.(18), we have

$$g[\{s_j\}] \simeq \frac{1}{\pi}\prod_j (2j+1)^{s_j}\left[\int_0^\pi e^{-\frac{1}{2}\alpha\xi^2} d\xi - \int_0^\pi e^{-\frac{1}{2}(4+\alpha)\xi^2} d\xi\right] \tag{19}$$

where we have used $\xi = \theta - \theta_0$ and the following limits

$$\lim_{\theta_0\to 0} \exp[G(\theta_0,k)] = \prod_j (2j+1)^{s_j}$$

$$\lim_{\theta_0\to 0} G''(\theta,k)|_{\theta_0} = -4\sum_j s_j\ j(j+1) \equiv -\alpha$$

$$\lim_{\theta_0\to 0} \sec^2 2\theta_0 = 1$$

Evaluation of the integral yields

$$g[\{s_j\}] \simeq \frac{1}{\sqrt{2\pi}}\prod_j (2j+1)^{s_j}\left(\sqrt{\frac{1}{\alpha}}\ \mathrm{Erf}\left[\pi\sqrt{\alpha/2}\right] - \sqrt{\frac{1}{4+\alpha}}\ \mathrm{Erf}\left[\pi\sqrt{(4+\alpha)/2}\right]\right) \tag{20}$$

The quantity α results from the second order approximation. Hence, while calculating α we can only use the results up to first order i.e. we use $s_j^\star$ in place of s_j whose expression will be given by

$$s_j^\star \simeq N(2j+1)\exp[-\lambda\sqrt{j(j+1)} - \sigma] \tag{21}$$

which follows from the fact that $g[\{s_j\}] \simeq \frac{2C}{\pi}\prod_j (2j+1)^{s_j}$ neglecting the second order corrections, C being some constant. Now, using eq.(21) in eq.(12a) and eq.(12b), one obtains

$$\exp[\sigma] = \sum_j (2j+1)\exp[-\lambda\sqrt{j(j+1)}] \tag{22a}$$

$$k/2 = N\sum_j \sqrt{j(j+1)}(2j+1)\exp[-\lambda\sqrt{j(j+1)} - \sigma] \tag{22b}$$

Using eq.(22a) and eq.(22b) one can show that $k = -2N(d\sigma/d\lambda)$ and consequently $\alpha \simeq 8N(d^2\sigma/d\lambda^2)$. Now, the quantity $\alpha \to \infty$ for $N \to \infty$. Hence, we can approximately write $\mathrm{Erf}\left[\pi\sqrt{(4+\alpha)/2}\right] \approx \mathrm{Erf}\left[\pi\sqrt{\alpha/2}\right]$. Plotting the function $\mathrm{Erf}\left[\pi\sqrt{\alpha/2}\right]$ with α one can see that the function attains a constant value for large α. Hence, we can take $\lim_{\alpha\to\infty} \mathrm{Erf}\left[\pi\sqrt{(4+\alpha)/2}\right] \simeq \lim_{\alpha\to\infty} \mathrm{Erf}\left[\pi\sqrt{\alpha/2}\right] \simeq K$, some constant. Therefore, from eq.(20) it follows that

$$\begin{aligned} g[\{s_j\}] &\simeq \frac{K}{\sqrt{2\pi}}\prod_j (2j+1)^{s_j}\left(\sqrt{\frac{1}{\alpha}} - \sqrt{\frac{1}{4+\alpha}}\right) \\ &\simeq K\sqrt{\frac{2}{\pi}}\prod_j (2j+1)^{s_j}\alpha^{-\frac{3}{2}} \end{aligned}$$

Therefore, we have

$$\frac{\delta}{\delta s_j}\log g[\{s_j\}] = \log(2j+1) - \frac{6}{\alpha}j(j+1) \tag{23}$$

Hence, considering variation of α resulting from the inclusion of the quadratic fluctuations, the distribution for the most probable configuration given by eq.(21) gets modified into

$$s_j^\star \simeq N(2j+1)\exp[-\lambda\sqrt{j(j+1)} - \sigma - \frac{6}{\alpha}j(j+1)] \tag{24}$$

Now, using the most probable distribution given by eq.(21) or eq.(24) (result will differ by a constant only) we calculate the microcanonical entropy of a QIH for given values of k and N using Stirling approximation and the result comes out to be

$$S_{MC} = \frac{\lambda k}{2} + N\sigma - \frac{3}{2}\log N - \frac{3}{2}\log(d^2\sigma/d\lambda^2) + \cdots \tag{25}$$

where λ and σ are explicit functions of k/N which can be obtained by solving equations (22a) and (22b)[22]. Now, for a QIH (i.e. at equilibrium), A_{cl} and N are related by the equation $A_{cl} = -8\pi\gamma\ell_p^2 N(d\sigma/d\lambda)$ and also we have $k = A_{cl}/4\pi\gamma\ell_p^2$. Hence, the microcanonical entropy can be expressed completely in terms of the classical area of the IH or the mean area of the QIH as

$$S_{MC}(A_{cl}) = \left[\frac{f(\lambda)}{2\pi\gamma}\right]\frac{A_{cl}}{4\ell_p^2} - \frac{3}{2}\log\frac{A_{cl}}{4\ell_p^2} + \cdots \tag{26}$$

where $f(\lambda) = \lambda - \left(\sigma/\frac{d\sigma}{d\lambda}\right)$. To retrieve the Bekenstein-Hawking area law(BHAL), one should fit $\gamma = f(\lambda)/2\pi$. One should note that, in this process, the value of γ becomes known in terms of k and N as λ is completely solvable in terms of those parameters from the equations (22a) and (22b). This issue has been explicitly discussed and shown in [22], although there is a subtle difference in the fitting of γ as far as the present work is concerned.

Now we calculate σ explicitly in the limit $k \to \infty$ to study the nature of the function $f(\lambda)$. We have

$$\lim_{k\to\infty}\exp[\sigma] \simeq \int_0^\infty (2j+1)\exp\left[-\lambda\sqrt{j(j+1)}\right]dj - 1 \tag{27}$$

which yields $\sigma = \log\left(\frac{2}{\lambda^2} - 1\right)$ and $d\sigma/d\lambda = 4/[\lambda(\lambda^2-2)]$. Hence, it follows after a few steps of algebra

$$f(\lambda) = \lambda\left[1 - \frac{1}{4}(2-\lambda^2)\log\left(\frac{\lambda^2}{2-\lambda^2}\right)\right] \tag{28}$$

Now, from information theoretic arguments it can be explained that $\sigma < 0$ [22], which in turn implies that $\lambda > 1$. On the other hand, for $f(\lambda)$ to be real (for positive entropy), we must have $\lambda^2 < 2$. Thus the allowed (positive) values of the parameter λ get restricted in the range $1 < \lambda < \sqrt{2}$. Also, one can check that the

corresponding range of the function $f(\lambda)$ is also given by $1 < f(\lambda) < \sqrt{2}$. Thus, BHAL is valid for a certain range of γ which is same as what has been reported in [22] i.e. $0.159 < \gamma < 0.225$. Hence, for this particular range of the Barbero-Immirzi parameter (γ), we have the microcanonical entropy of an IH given by

$$S_{MC} = S_{BH} - \frac{3}{2} \log S_{BH} + \cdots \tag{29}$$

where $S_{BH} = A_{cl}/4\ell_p^2$ is the Bekenstein-Hawking entropy and the smaller terms are neglected. From eq.(29), one can clearly observe that the universal coefficient in front of the logarithmic correction is $-3/2$ as usual[23, 28]. But, from eq.(25), it is revealed that while expressed in terms of k and N, there is a fundamental effect due to the presence of the topological defects, the punctures, playing the role of a new 'quantum hair' first proposed in [19]. Also we have an appropriate bound on the permissible values of γ, considering the BHAL as an external input as pointed out in [22].

V. DISCUSSION

As mentioned in the Introduction, the LQG formulation of a QIH in terms of a CS theory coupled to spins directly involves the use of the quantum parameters ('hairs') k and N. Strictly speaking, if the classical area A_{cl} is taken to be a hair, the Barbero-Immirzi parameter γ (the coefficient of a topological contribution to the classical action from the Nieh-Yan invariant [31]) is considered as an independent coupling parameter, then the only new parameter that has appeared in the quantum theory is the number of punctures N. Equivalently, one can, as we have in this paper, take k and N to be the parameters characterising the quantum theory. It is significant that the requirement that the microcanonical entropy of a QIH yields the area law for large k , N immediately restricts the Barbero-Immirzi parameter γ to lie in a specific interval on the real line, while also yielding the subleading logarithmic correction derived in earlier literature with the universal coefficient $-3/2$. Thus, the complete characterization of a QIH is given by two independent parameters, namely, k and N.

The departure from recent literature is the direct link established in this paper of these results and interpretation within the CS formulation of a QIH. The point here is that the computation of the microcanonical entropy of a QIH is *not* merely a combinatoric exercise in statistical mechanics of an ideal gas of punctures with some restrictions gleaned out of LQG as in some part of the recent literature [19, 32–37], with at best a weak link to the complete formulation as a CS theory that has been available for years. Rather, the original formulation yields an understanding that is complete as a quantum theory.

An extension of the foregoing analysis to Trapping horizons and Entanglement entropy of matter and radiation fields in their vicinity, would amount to truly new physics of radiant quantum horizons beyond general relativity. This has within it the potential to surpass, because of its firmer quantum geometric underpinning, recently proposed speculative ideas of a somewhat ad hoc nature based on semiclassical analysis [38, 39], about how classical horizon geometry must change (become a 'fuzzball' or a 'firewall') so as to allow the existence of well-defined scattering amplitudes for quantum matter fields. Whether or not such a structure emerges from LQG in an appropriate limit is not known at the moment, since there is still no complete quantum geometric analysis of Hawking radiation and its various conundra from an LQG standpoint. The problem at hand involves a quantization of the kinematical phase space of a Trapping/Dynamical horizon along the lines of [1, 2] which may not be technically so simple as a QIH because of the transient behaviour inherent in the geometry. However, perhaps a perturbation of the CS description by a set of appropriately chosen matter field operators might serve as a first approximation to the problem. The strength of the LQG approach lies in the transparent manner in which the CS symplectic structure emerges from the boundary conditions in the incipient formulation [2], without having to rely on conjectured results. The relation of the CS Hilbert space to the conformal blocks of the WZW model on the QIH is also not a matter of conjecture for large black holes. Thus, the LQG analysis of a QIH geometry throws up holographic structures as *emergent*, without any prior notion that they have to be there. One expects that generalizations to Trapping horizons to centre around this theme so as to reap the benefits of the 4 dimensional gravity - 2 dimensional conformal field theory link found in ref. [20]. Note however that the thermal stability of radiant trapping horizons which approach an equilibrium QIH has been discussed within the LQG framework yielding a criterion of stability involving the equilibrium mass and the microcanonical

entropy of the QIH in ref. [40–42].

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